Weak Convergence to Stochastic Integrals Driven by $\alpha-$Stable Lévy Processes *

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Abstract: We use the martingale convergence method to get the weak convergence theorem on general functionals of partial sums of independent heavy-tailed random variables. The limiting process is the stochastic integral driven by $\alpha-$stable Lévy process. Our method is very powerful to obtain the limit behavior of heavy-tailed random variables.

Keyword: Weak convergence, martingale convergence, stochastic integral, $\alpha-$stable Lévy process, heavy-tailed.

1 Introduction

Let $X_n, n \geq 1$, be independent and identically distributed (i.i.d.) random variables. When the distribution of X_1 is heavy-tailed, the limit behavior of stochastic processes which are related to $\{X_n\}$ are very important and interesting. In this paper, we will discuss the weak convergence of following processes:

$$\sum_{i=2}^{[nt]} f(\sum_{j=1}^{i-1}(X_{n,j} - E(h(X_{n,j})))(X_{n,i} - E(h(X_{n,i}))), \tag{1.1}$$

where $X_{n,j} = X_j/b_n$ for some $b_n \to \infty$, $f(x), h(x)$ are continuous functions.

This type of limit theorems is very important in probability theory, mathematical statistics and econometrics, especially, it is a core theory in the unit root model, which is a hot topic in the econometric theory (c.f. Phillips (1987 a,b), (2007)). In the unit root theory, the limiting process of stochastic process sequence like (1.1) is a stochastic integral. In Ibragimov and Phillips (2008), they studied the weak convergence of stochastic processes like (1.1) when $X_n, n \geq 1$, are linear processes. Their theorems are extension of unit root results. Lin and Wang (2010) studied the same problems for causal processes.

In this paper, we extend these results to the heavy-tailed random variables. Heavy-tailed analysis is an interesting and important branch of probability, stochastic process and mathematical statistics. Record-breaking insurance losses, financial log-returns, transmission rates of files are examples of

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heavy-tailed phenomena. According to Rvačeva (1962), if the $X_j,\ j \geq 1$, are i.i.d., there exist $b_n > 0$ and c_n such that

$$\frac{1}{b_n}\sum_{j=1}^{n} X_j - c_n \xrightarrow{d} \varsigma_\alpha \tag{1.2}$$

for some non-degenerate $\alpha-$stable random variable ς_α with $\alpha \in (0,2)$ if and only if X_1 is regularly varying with index $\alpha \in (0,2)$. After then, a lot of authors studied the asymptotic behavior of independent or dependent heavy-tailed random variables. A detailed study of conditions for convergence of the partial sums of dependent stationary process to an infinite variance stable distribution was given in Bartkiewicz, Jakubowski, Mikosch and Wintenberger (2010), they also gave s survey for asymptotic distribution of partial sums of dependence heavy-tailed random variables.

An extension of (1.2) is its functional version for partial sum processes. In the other words, we consider the following processes:

$$X_n(t) = \frac{1}{b_n}\sum_{i=1}^{[nt]-1} X_i - tc_n, \tag{1.3}$$

where $X_n(\cdot)$ is a random elements with values in the Skorohod space $\mathbb{D}[0,1]$, i.e., the space of all function on $[0,1]$ that are right-continuous and have left limits. Then the weak convergence of (1.3) is that for the probability measure on the space $\mathbb{D}[0,1]$. Many authors discussed this convergence. The point process method is a very powerful method to obtain this type of weak convergence. This method was given in detailed by Resnick(1986). They combined the weak convergence of point processes with the continuous mapping theorem to obtain the results. They showed that that the limiting process of (1.3) is $\alpha-$stable Lévy process if X_1 is regularly varying with index $\alpha \in (0,2)$. Davis and Hsing (1995) extends this result to dependent case.

In this paper, we will discuss the weak convergence of stochastic processes (1,1). In fact, they can be seen as the discretizations of stochastic integrals. When X_1 is regularly varying with index $\alpha \in (0,2)$, we get that the limiting process of (1.1) is a stochastic integral driven by $\alpha-$stable Lévy process.

The weak convergence of (1.1) is interesting and difficulty from the theoretical point. If we use the point process method to obtain the weak convergence, the summation functional should be proved as a continuous functional respect to the topology of Skorohod space $\mathbb{D}[0,1]$, and the limiting process should have a compound Poisson representation. However, the summation functional like (1.1) is difficult to be proved as a continuous functional in the Skorohod space $\mathbb{D}[0,1]$. Moreover, the stochastic integral driven by $\alpha-$stable Lévy process don't have a compound Poisson representation. The point process method can not be used easily.

In this paper, we will use the stochastic calculus method to obtain the result. Since the limiting process is a semimartingale, we will use the predictable characteristics of semimartingale to describe the asymptotic behavior of underlying processes. This is a very common method in the study of classical stochastic analysis. More details can be found in Jacod and Shiryaev (2003), which introduces predicable characteristics to replace the three terms in the usual case: the drift, variance of the Guassian part and the Lévy measure, which characterize the distribution of the Lévy process. By means of these three characteristics, the tightness criteria of semimartingale sequence is obtained. Furthermore, one can identify the law of limiting process through the unique solution of martingale problem related to the predicable characteristics. In some special cases, the unique solution of a

martingale problem can be seen as a unique solution of stochastic differential equations (for example, when the limiting process is a stochastic integral). In this paper, we firstly compute the predicable characteristics of stochastic integral, and then we through the so-called martingale convergence method to get the criteria conditions for weak convergence. The martingale convergence method is also summarized in Jacod and Shiryaev (2003). This method is based on the martingale characteristic of semimartingale. When the limiting process is a semimartingale, martingale convergence method is very powerful.

The assumptions for obtaining main results are same as those in the point process method, our method is more simple than the point process method. In Jacod and Shiryaev (2003), the authors use a same truncate function to get the special semimartingale, and the predicable characteristics of special martingale can determine the limit behavior of semimartingale. However, we use a truncate function to get the special semimartingales for (1.1), and employ another different truncate function to deal with the limiting process. We employ the core idea of the martingale convergence method to show the result. Our method is a modification of the martingale convergence method. It is more convenience to verify the tightness conditions. As we know, the stochastic calculus method and martingale convergence method are not used in the asymptotic analysis of heavy-tailed phenomena in the previous study, our method may be a new complement to the study of heavy-tailed phenomena. The similar procedure was used in our previous paper, Lin and Wang (2010). Since the limiting process in that paper has no jumps, it is more simple than that in this paper. However, the jumps in the limiting process play a major role in the asymptotic analysis.

The remainder of this paper is organized as follows. Section 2 collects some basic tolls and notations to be used throughout this paper. In Section 3 and Section 4, we discuss the weak convergence to stochastic integrals driven by stable processes in the univariate and multivariate case respectively. Some discussion about the further research is given in Section 5.

2 Preliminary

2.1 Predictable Characteristics of Semimartingale and Convergence of Semimartingales.

We follow the semimartingale theory as presented in Jacod and Shiryaev (2003). For our purpose, let $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$. $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 1}, P)$ is a filtered probability space. X is a semimartingale defined on $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 1}, P)$. Set $h(x)$ is a continuous function satisfying $h(x) = x$ in a neighbourhood of 0 and $|h(x)| \leq |x|1_{|x|\leq 1}$. Let

$$\begin{cases} \check{X}(h)_t = \sum_{s\leq t}[\Delta X_s - h(\Delta X_s)], \\ X(h) = X - \check{X}(h), \end{cases}$$

where $\Delta X_s = X_s - X_{s-}$. $X(h)$ is a special semimartingale and we consider its canonical decomposition:

$$X(h) = X_0 + M(h) + B(h), \tag{2.1}$$

where $M(h)$ is its local martingale part, $B(h)$ is its finite variation part.

Definition 1 (Jacod and Shiryaev (2003)) We call predictable characteristics of X the triplet (B, C, ν) as follows:

(1) B is a predictable finite variation process, namely the process $B = B(h)$ appearing in (2.1).

(2) $C =< M(h), M(h) >$ is a predictable process.

(3) ν is a predictable random measure on $\mathbb{R}_+ \times \mathbb{R}$, namely the compensator of the random measure μ^X associated to the jumps of X, μ^X is defined by

$$\mu^X(\omega; dt, dx) = \sum_s 1_{\{\triangle X_s(\omega) \neq 0\}} \varepsilon_{(s, \triangle X_s(\omega))}(dt, dx), \tag{2.2}$$

where ε_a denotes the Dirac measure at the point a, which may be from different spaces.

Definition 2 (Jacod and Shiryaev (2003)) Let X be a càdlàg process and let $\mathcal{H}$ be the $\sigma-$field generated by $X(0)$ and $\mathcal{L}_0$ be the distribution of $X(0)$. A solution to the martingale problem associated with $(\mathcal{H}, X)$ and $(\mathcal{L}_0, B, C, \nu)$ (denoted by $\varsigma(\sigma(X_0), X|\mathcal{L}_0, B, C, \nu))$ is a probability measure P on $(\Omega, \mathscr{F})$ such that X is a semimartingale on $(\Omega, \mathscr{F}, P)$ with predictable characteristics (B, C, ν).

The limit process $X = (X(s))_{s\geq 0}$ appearing in this paper is the canonical process $X(s, \alpha) = \alpha(s)$ for the element $\alpha = (\alpha(s))_{s\geq 0}$ of $D([0,1])$. In other words, our limit process is defined on the canonical space $(\mathbb{D}([0,1]), \mathscr{D}([0,1]), \mathbf{D})$. For $a \geq 0$ and an element $(\alpha(s), s \geq 0)$ of the Skorokhod space $\mathbb{D}([0,1])$, define

$$S^a(\alpha) = \inf(s : |\alpha(s)| \geq a \text{ or } |\alpha(s-)| \geq a).$$

In the paper, $\Rightarrow$ denotes weak convergence in an appropriate metric space, and $\stackrel{P}{\longrightarrow}$ denotes convergence in probability. $\mathbb{C}_2^b(R)$ denotes the set of all bounded continuous functions on $\mathbb{R}$ which are 0 around 0. $\mathbb{C}_1^b(R)$ is a subclass of $\mathbb{C}_2^b(R)$ having only nonnegative functions, which contains all functions $g_a(x) = (a|x|-1)^+ \wedge 1$ for all positive rationals a and it is a convergence-determining class for the weak convergence induced by $\mathbb{C}_2^b(R)$. For a finite variation process A, the total variation process of A is denoted by $Var(A)$. For K and H, $K \cdot H$ denotes the stochastic integral. The following propositions, provides the basis for the study of asymptotic properties of semimartingales, they can be found in Jacod and Shiryaev (2003).

Proposition A Let $\{X^n, n \geq 1\}$ be a sequence of càdlàg processes, and suppose that for all $n, q \in N$, we have the decomposition

$$X^n = U^{nq} + V^{nq} + W^{nq}$$

satisfying that

(i) the sequence $(U^{nq})_n$ is tight;

(ii) the sequence $(V^{nq})_n$ is tight and there is a sequence (a_q) of real numbers such that:

$$\lim_{q\to\infty} a_q = 0, \quad \lim_{n\to\infty} P(\sup_{t\leq 1} |\Delta V^{nq}| > a_q) = 0;$$

(iii) for $\varepsilon > 0$,

$$\lim_{q\to\infty} \limsup_{n\to\infty} P(\sup_{t\leq 1} |W^{nq}| > \varepsilon) = 0.$$

Then the sequence (X^n) is tight.

Proposition B Let Y^n be a càdlàg process and M^n be a martingale on a same filtered probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 1}, P)$. Let M be a càdlàg process defined on the canonical space $(\mathbb{D}([0,1]), \mathscr{D}([0,1]), \mathbf{D})$. Assume that

(i) (M^n) is uniformly integrable;

(ii) $Y^n \Rightarrow Y$ for some Y with law $\widetilde{P} = \mathscr{L}(Y)$;

(iii)

$$M_t^n - M_t \circ (Y^n) \xrightarrow{P}, \quad 0 \le t \le 1$$

Then the process $M \circ (Y)$ is a martingale under $\widetilde{P}$.

2.2 Heavy-tailed Random Variable and Lévy $\alpha-$Stable Process

In this subsection, we collect some facts, tools and notions about heavy-tailed random variables. Roughly speaking, a random variable X heavy-tailed with index $\alpha \in (0,2)$ if there exists a positive parameter α such that

$$P(X > x) \sim x^{-\alpha}, \quad x \to \infty.$$

Usually, people discuss a class of heavy-tailed random variables, the so-called stable random variables which will also be discussed in this paper.

Definition 3 A random variable X is said to be α-stable if its characteristic function is given by

$$E\mathsf{exp}\{iuX\} = \mathsf{exp}\{iua_\alpha + \int (e^{iux} - 1 - iuh(x))\Pi_\alpha(dx)\},$$

$$a_\alpha = \begin{cases} \beta\frac{\alpha}{1-\alpha}, & \alpha \neq 1, \\ 0, & \alpha = 1, \end{cases}$$

the index of stability $\alpha \in (0,2)$ and $\Pi_\alpha(dx)$ is the Lévy measure.

In this paper, we use vague convergence to be assumption. Some backgrounds on vague convergence are given below. More details can be found in Resnick (2007). More details can be found in Resnick (2007). Let E be a locally compact Hausdorff space with a countable basis and $M_p(E)$ be the set of Radon measures on E with values in $\mathbb{Z}_+$, where $\mathbb{Z}_+$ denotes the set of positive integers. The space $M_p(E)$ is a Polish space which is endowed with the topology of vague convergence. Recall that for $\mu_n, \mu \in M_p(E)$

$$\mu_n \xrightarrow{v} \mu \ \text{ iff } \ \mu_n(f) \to \mu(f)$$

for any $f \in C_K^+$, where C_K^+ is the class of continuous functions with compact support. In this paper, we assume $E = [-\infty, \infty]\backslash\{0\}$.

The stochastic integral, which will be considered, is driven by $\alpha-$stable Lévy process. It is a pure-jump process, in the other words, it can be presented as a point process. Let $X_\alpha(t)$ be a $\alpha-$stable Lévy Process, $\Delta X_\alpha(t) = X_\alpha(t) - X_\alpha(t-)$, and

$$\mu_\alpha(dt, dx) := \mu_\alpha(\omega, dt, dx) = \sum_s 1_{\{\triangle X_\alpha(s)(\omega) \neq 0\}} \varepsilon_{(s, \triangle X_\alpha(s)(\omega))}(dt, dx). \tag{2.3}$$

We assume the predictable compensator of $\mu_\alpha(dt, dx)$ is $ds\nu(dx)$, where $\nu(dx)$ is the Lévy measure of $X_\alpha(1)$. By the Lévy-Itô representation of Lévy process,

$$X_\alpha(t) = \int_0^t \int h(x)(\mu_\alpha(ds, dx) - ds\nu(dx)) + \int_0^t \int (x - h(x))\mu_\alpha(ds, dx), \tag{2.4}$$

where $h(x)$ is a continuous truncate function.

3 Convergence to Stochastic Integral Driven by a Lévy $\alpha-$Stable Process: The Univariate Case.

In this section, we use the martingale convergence approach to obtain the weak convergence results for various general functionals of partial sums of i.i.d. heavy-tailed random variables. The method of proof for these results is new in the study of heavy-tailed analysis.

Our main results are about the weak convergence of stochastic processes in $\mathbb{D}[0,1]$ with Skorohod J_1 topology.

Theorem 1 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous differentiable function such that

$$|f(x) - f(y)| \leq K|x - y|^a \tag{3.1}$$

for some constants $K > 0$, $a > 0$ and all $x, y \in \mathbb{R}$. Suppose that $\{X_n\}_{n\geq 1}$ is a sequence of i.i.d. random variables. Set

$$X_{n,j} = \frac{X_j}{b_n} - E(h(\frac{X_j}{b_n})) \tag{3.2}$$

for some $b_n \to \infty$. Define ρ by

$$\rho((x, +\infty]) = px^{-\alpha}, \quad \rho([-\infty, -x)) = qx^{-\alpha} \tag{3.3}$$

for $x > 0$, where $\alpha \in (0,1)$, $0 < p < 1$ and $p + q = 1$. Then

$$(\sum_{i=1}^{[nt]} X_{n,i}, \sum_{i=2}^{[nt]} f(\sum_{j=1}^{i-1} X_{n,j})X_{n,i}) \Rightarrow (Z_\alpha(t), \int_0^t f(Z_\alpha(s-))dZ_\alpha(s)), \tag{3.4}$$

in $\mathbb{D}[0,1]$, where $Z_\alpha(s)$ is an $\alpha-$stable Lévy process with Lévy measure ρ iff

$$nP[\frac{X_1}{b_n} \in \cdot] \xrightarrow{v} \rho(\cdot) \tag{3.5}$$

in $M_p(E)$.

Theorem 2 Let function f be same as that in Theorem 1, and $\{X_n\}_{n\geq 1}$ is a sequence of i.i.d. random variables. Set

$$X^\varepsilon_{n,j} = \frac{X_j}{b_n} 1_{\{|X_j|\geq\varepsilon\}} - E(h(\frac{X_j}{b_n} 1_{\{|X_j|\geq\varepsilon\}}))$$

and

$$Z^\varepsilon_\alpha(t) = \int_0^t \int_{|x|>\varepsilon} h(x)(\mu(ds, dx) - ds\nu(dx)) + \int_0^t \int (x - h(x))\mu(ds, dx)$$

for any $\varepsilon > 0$ and some $b_n \to \infty$. Define ρ as (3.3) for $\alpha \in [1,2)$, Then

$$(\sum_{i=1}^{[nt]} X^\varepsilon_{n,i}, \sum_{i=2}^{[nt]} f(\sum_{j=1}^{i-1} X^\varepsilon_{n,j})X^\varepsilon_{n,i}) \Rightarrow (Z^\varepsilon_\alpha(t), \int_0^t f(Z^\varepsilon_\alpha(s-))dZ^\varepsilon_\alpha(s)),$$

iff (3.5) stands.

Remark 1. Usually, X_1 is assumed to be mean zero and symmetric random variable, but we don't have such assumptions. It can not be deal with easily through centralized and symmetric procedure. When $\sum_{i=1}^{[nt]} X_{n,i}$ is integrator of integral,

$$\sum_{i=2}^{[nt]} f(\sum_{j=1}^{i-1} X_{n,j})(\sum_{k=1}^{i} X_{n,k} - \sum_{k=1}^{i-1} X_{n,k}),$$

the mean part produces another stochastic processes through $f(\sum_{j=1}^{[nt]-1}X_{n,j})$,

$$\sum_{i=2}^{[nt]} f(\sum_{j=1}^{i-1} X_{n,j})E(h(\frac{X_j}{b_n})).$$

After centralized and symmetric procedure, the limiting process of weak convergence maybe change.

Remark 2. When $\alpha \in [1,2)$,

$$\int_0^1 x\rho(dx) = \infty.$$

it is different from the case of $\alpha \in (0,1)$ and is more difficult to obtain the same result as Theorem 1. We obtain a weaker result.

Set

$$Y_n(t) = \sum_{i=2}^{[nt]} f(\sum_{j=1}^{i-1} X_{n,j})X_{n,i}, \quad Y(t) = \int_0^t f(Z_\alpha(s-))dZ_\alpha(s), \quad S_n(t) = \sum_{i=1}^{[nt]} X_{n,i}.$$

We want to prove

$$H_n(t) := (Y_n(t), S_n(t)) \Rightarrow H(t) = (Y(t), Z_\alpha(t)).$$

We firstly give some lemmas, which are the basis of the proof.

Lemma 1. The predictable characteristics of $(Y(t), Z_\alpha(t))$ are the terms (B, C, λ) as follows:

$$\begin{cases} B^i(t) = \begin{cases} \int_0^t \int (h(f(Z_\alpha(s-)x) - f(Z_\alpha(s-)h(x))\nu(ds,dx), & i=1, \\ 0, & i=2, \end{cases} \\ C^{ij}(t) = \begin{cases} \int_0^t \int h^2(f(Z_\alpha(s-)x)\nu(ds,dx), & i=1, j=1, \\ \int_0^t \int h(f(Z_\alpha(s-)x)h(x)\nu_n(ds,dx), i=1, j=2, \text{or } & i=2, j=1, \\ \int_0^t \int h^2(x)\nu(ds,dx), & i=2, j=2, \end{cases} \\ 1_G * \lambda(ds,dx) = 1_G(x, f(Z_\alpha(s-))x)\nu(ds,dx) \text{ for all } G \in \mathbb{B}^2, \end{cases}$$

where $\nu(ds,dx)$ is the compensator of the jump measure of $Z_\alpha(t)$.

Proof. From (2.4) and $\nu(\{t\}\times dx) = 0$, $Z_\alpha(t)$, $B(t)$ and $C^{22}(t)$ are obtained by Proposition 2.17 in Chapter 2 of Jacod and Shiryaev(2003).

Let $\eta(ds,dx)$ be the jump random measure of $Y(t)$ and $\lambda'(ds,dx)$ be the compensator of $\eta(ds,dx)$.

If G is a Borel set in R, we have

$$1_G * \lambda'(ds,dx) = 1_G(f(Z_\alpha(s-))x) * \nu(ds,dx).$$

Set $z = f(Z_\alpha(s-))x$, then

$$\begin{aligned} & Y(t) - \int_0^t \int (z - h(z))\eta(ds,dz) \\ = & \int_0^t \int f(Z_\alpha(s-))h(x)(\mu(ds,dx) - \nu(ds,dx)) + \int_0^t \int f(Z_\alpha(s-))(x-h(x))\mu(ds,dx) \\ & - \int_0^t \int (z-h(z))\eta(ds,dz) \\ = & \int_0^t \int f(Z_\alpha(s-))h(x)(\mu(ds,dx) - \nu(ds,dx)) + \int_0^t \int f(Z_\alpha(s-))(x-h(x))\mu(ds,dx) \\ & - \int_0^t \int (f(Z_\alpha(s-))x - h(f(Z_\alpha(s-)x))\mu(ds,dx) \end{aligned}$$

$$\begin{aligned}
&= \int_0^t \int f(Z_\alpha(s-))h(x)(\mu(ds,dx) - \nu(ds,dx)) \\
&\quad + \int_0^t \int (h(f(Z_\alpha(s-)x) - f(Z_\alpha(s-)h(x))\mu(ds,dx) \\
&= \int_0^t \int f(Z_\alpha(s-))h(x)(\mu(ds,dx) - \nu(ds,dx)) \\
&\quad + \int_0^t \int h(f(Z_\alpha(s-))x) - f(Z_\alpha(s-))h(x)(\mu(ds,dx) - \nu(ds,dx)) \\
&\quad + \int_0^t \int h(f(Z_\alpha(s-))x) - f(Z_\alpha(s-))h(x)\nu(ds,dx) \\
&= \int_0^t \int h(f(Z_\alpha(s-))x)(\mu(ds,dx) - \nu(ds,dx)) \\
&\quad + \int_0^t \int h(f(Z_\alpha(s-))x) - f(Z_\alpha(s-))h(x)\nu(ds,dx),
\end{aligned}$$

which implies

$$B_t^1 = \int_0^t \int h(f(Z_\alpha(s-))x) - f(Z_\alpha(s-))h(x)\nu(ds,dx),$$

and the martingale part of Y_t is

$$\int_0^t \int h(f(Z_\alpha(s-))x)(\mu(ds,dx) - \nu(ds,dx)). \tag{3.6}$$

Then we can get C^{11}, C^{12} and C^{21}. The lemma is proved. ■

We set

$$\mu_n(\omega; ds, dx) = \sum_{i=1}^n \varepsilon_{(\frac{i}{n}, \frac{X_i(\omega)}{b_n})}(ds, dx),$$

then

$$\nu_n(\omega; ds, dx) := \sum_{i=1}^n \varepsilon_{(\frac{i}{n})}(ds) P(\frac{X_i}{b_n} \in dx)$$

is the compensator of μ_n by the independent of $\{X_i\}_{i\geq 1}$. Set

$$\zeta_n(\omega; ds, dx) = \sum_{i=1}^n \varepsilon_{(\frac{i}{n}, \frac{X_i(\omega)}{b_n} - c_n)}(ds, dx),$$

we have

$$\varphi_n(\omega; ds, dx) := \sum_{i=1}^n \varepsilon_{(\frac{i}{n})}(ds) P(\frac{X_i}{b_n} - c_n \in dx)$$

is the compensator of $\zeta_n(\omega; ds, dx)$, where $c_n = E[h(\frac{X_1}{b_n})]$.

Firstly, we consider process $S_n(t)$. Introduce truncate function $h_n(x) = h(x + c_n)$.

$$\begin{aligned}
S_n(t) &= \sum_{i=1}^{[nt]} h(\frac{X_i}{b_n}) + \sum_{i=1}^{[nt]} (X_{n,i} - h(\frac{X_i}{b_n})) \\
&= \sum_{i=1}^{[nt]} (h(\frac{X_i}{b_n}) - c_n) + \sum_{i=1}^{[nt]} (\frac{X_i}{b_n} - h(\frac{X_i}{b_n}))
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \int h(x)(\mu_n(ds,dx)-\nu_n(ds,dx)) + \sum_{i=1}^{[nt]}(\frac{X_i}{b_n} - h(\frac{X_i}{b_n}))\\
&=: \widetilde{S}_n(t) + \sum_{i=1}^{[nt]}(\frac{X_i}{b_n} - h(\frac{X_i}{b_n})).
\end{aligned}
$$

The predictable characteristics of $\widetilde{S}_n(t)$ are

$$
B_n^2(t) = 0,
$$

$$
C_n^{22}(t) = \int_0^t \int h^2(x)\nu_n(ds,dx) - \sum_{s\le t}(\int h(x)\nu_n(\{s\},dx))^2.
$$

For $Y_n(t)$, we have

$$
\begin{aligned}
Y_n(t) &= \sum_{i=2}^{[nt]} h(f(\sum_{j=1}^{i-1} X_{n,j})\frac{X_i}{b_n}) + \sum_{i=2}^{[nt]}(f(\sum_{j=1}^{i-1} X_{n,j})X_{n,i} - h(f(\sum_{j=1}^{i-1} X_{n,j})\frac{X_i}{b_n}))\\
&= \sum_{i=2}^{[nt]}(h(f(\sum_{j=1}^{i-1} X_{n,j})\frac{X_i}{b_n}) - E(h(f(\sum_{j=1}^{i-1} X_{n,j})\frac{X_i}{b_n})|\mathscr{F}_i))\\
&\quad + \sum_{i=2}^{[nt]}(E(h(f(\sum_{j=1}^{i-1} X_{n,j})\frac{X_i}{b_n})|\mathscr{F}_i) - f(\sum_{j=1}^{i-1} X_{n,j})E(h(\frac{X_1}{b_n})))\\
&\quad + \sum_{i=2}^{[nt]}(f(\sum_{j=1}^{i-1} X_{n,j})\frac{X_i}{b_n} - h(f(\sum_{j=1}^{i-1} X_{n,j})\frac{X_i}{b_n}))\\
&= \int_0^t \int h(f(\sum_{j=1}^{[ns]-1} X_{n,j})x)(\mu_n(ds,dx)-\nu_n(ds,dx))\\
&\quad + \int_0^t \int (h(f(\sum_{j=1}^{[ns]-1} X_{n,j})x) - f(\sum_{j=1}^{[ns]-1} X_{n,j})h(x))\nu_n(ds,dx)\\
&\quad + \sum_{i=2}^{[nt]}(f(\sum_{j=1}^{i-1} X_{n,j})\frac{X_i}{b_n} - h(f(\sum_{j=1}^{i-1} X_{n,j})\frac{X_i}{b_n}))\\
&=: \widetilde{Y}_n(t) + \sum_{i=2}^{[nt]}(f(\sum_{j=1}^{i-1} X_{n,j})\frac{X_i}{b_n} - h(f(\sum_{j=1}^{i-1} X_{n,j})\frac{X_i}{b_n})).
\end{aligned}
$$

The predictable characteristics of $\widetilde{Y}_n(t)$ are

$$
B_n^1(t) = \int_0^t \int (h(f(\sum_{j=1}^{[ns]-1} X_{n,j})x) - f(\sum_{j=1}^{[ns]-1} X_{n,j})h(x))\nu_n(ds,dx),
$$

$$
C_n^{11}(t) = \int_0^t \int h^2(f(\sum_{j=1}^{[ns]-1} X_{n,j})x)\nu_n(ds,dx) - \sum_{s\le t}(\int h(f(\sum_{j=1}^{[ns]-1} X_{n,j})x)\nu_n(\{s\},dx))^2,
$$

$$
C_n^{12}(t) = C_n^{21}(t)
$$

$$= \int_0^t \int h(f(\sum_{j=1}^{[ns]-1} X_{n,j})x)h(x)\nu_n(ds,dx)$$
$$-\sum_{s\le t}(\int h(f(\sum_{j=1}^{[ns]-1} X_{n,j})x)\nu_n(\{s\},dx))(\int h(x)\nu_n(\{s\},dx)).$$

Lemma 2. Under (3.5),

$$\int g(x)nF_n(dx) \to \int g(x)\rho(dx), \qquad n\to\infty, \tag{3.7}$$

for every continuous $g \in \mathbb{C}_2^b(R)$,where $F_n(x) = P(\frac{X_1}{b_n} \le x)$.

Proof. From (3.5), we have

$$\int h(x)nF_n(dx) \to \int h(x)\rho(dx), \qquad n\to\infty, \tag{3.8}$$

for every continuous compact support function h.

From (3.3), we can get that for any $\varepsilon > 0$, there exists $r > 0$ such that $\rho((r,+\infty))+\rho((-\infty,-r)) < \varepsilon$.

Set $B_r = [-r, r]$, we can find a continuous, compact support function g_r, such that $1_{B_r} \le g_r \le 1$. Then

$$\begin{aligned}|\int g(x)nF_n(dx) - \int g(x)\rho(dx)| \ &\le\ |\int g(x)nF_n(dx) - \int g(x)g_r(x)nF_n(dx)| + |\int g(x)g_r(x)nF_n(dx) \\ &\quad - \int g(x)g_r(x)\rho(dx)| + |\int g(x)g_r(x)\rho(dx) - \int g(x)\rho(dx)| \\ &\le\ |\int g(x)g_r(x)nF_n(dx) - \int g(x)g_r(x)\rho(dx)| + ||g||(nF_n(B_r^c) + \rho(B_r^c)).\end{aligned}$$

For $\varepsilon > 0$, there exists n_0, such that as $n \ge n_0$,

$$|\int g(x)g_r(x)nF_n(dx) - \int g(x)g_r(x)\rho(dx)| < \varepsilon.$$

From Theorem 3.2 (ii) in Resnick(2007), there exists n_1, as $n \ge n_1$,

$$|nF_n(B_r^c) - \rho(B_r^c)| < \varepsilon.$$

Then we have

$$|\int g(x)nF_n(dx) - \int g(x)\rho(dx)| \le (3||g|| + 1)\varepsilon$$

as $n \ge \max\{n_0, n_1\}$, which implies (3.7). ■

From (3.5), we can obtain

$$\sum_{i=1}^{[nt]} X_{n,i} \Rightarrow Z_\alpha(t), \tag{3.9}$$

by Corollary 7.1 in Resnick (2007).

So $\sum_{i=1}^{[nt]} X_{n,i}$ is related compact, in the other words, $\sum_{i=1}^{[nt]} X_{n,i}$ is tightness.

By the tightness of $\sum_{i=1}^{[nt]} X_{n,i}$, we have that for any $\varepsilon > 0$, there are $n_0 \in N$ and $K \in R^+$ with

$$P(\sup_{t\le 1} |S_n(t)| > K) < \varepsilon \ \text{ as } \ n \ge n_0. \tag{3.10}$$

Since the convergence of $H_n(t) \Rightarrow H(t)$ is a local property, it suffices to prove the Theorem 1 and 2 for $f(S_n(t-))1_{[0,T]}$ and $f(Z_\alpha(t-))1_{[0,T]}$ for any stopping time T.

We use S_n^C and S^C to replace T in $f(S_n(t-))1_{[0,T]}$ and $f(Z_\alpha(t-))1_{[0,T]}$ respectively, where $S_n^C = \inf(s : |S_n(s)| \geq C$ or $|S_n(s-)| \geq C)$. As described in Pagès (1986), we can assume

$$f(S_n(t-)) \leq C, f(Z_\alpha(t-)) \leq C \tag{3.11}$$

identically for some constant C in the following proof.

Let $\mathscr{K}$ be a compact subset of R such that $|u| \leq C$ for any $u \in \mathscr{K}$.

Set

$$1_G * \lambda_n(ds, dx) = 1_G(x, f(\sum_{i=1}^{[ns]-1} X_{n,i})x)\nu_n(ds, dx) \text{ for } \ G \in \mathbb{B}^2.$$

Lemma 3 Under (3.5), we have that for $t > 0$,

$$Var[K * \lambda_n - (K * \lambda) \circ H_n]_t \xrightarrow{P} 0, \tag{3.12}$$

for every continuous $K(x, u) \in \mathbb{C}_2^b(R \times \mathscr{K})$ satisfying $K(x, u) = 0$ for all $|x| \leq \delta$, $u \in \mathscr{K}$ for some $\delta > 0$.

Proof. Since this lemma is almost same as the Lemma IX 5.22 of Jacod and Shiryaev(2003). We verify thatthe assumptions of Lemma IX 5.22 of Jacod and Shiryaev(2003) are satisfied.

At first, we show that for every continuous $g \in \mathbb{C}_2^b(R)$,

$$Var[g * \nu_n - g * \nu]_t \to 0 \quad \text{ for } t > 0, \tag{3.13}$$

which is assumption (i) of the Lemma IX5.22 in Jacod and Shiryaev(2003). In fact,

$$\int_0^t \int g(x)\nu_n(ds, dx) = [nt]E(g(X_{n,1})),$$

and

$$\int_0^t \int g(x)\nu(ds, dx) = t \int g(x)\rho(dx).$$

We have

$$Var[g * \nu_n - g * \nu]_t \leq |\int g(x)nF_n(dx) - \int g(x)\rho(dx)|\frac{[nt]}{n} + |\frac{[nt]}{n} - t| \int g(x)\rho(dx),$$

(3.13) is obtained by (3.7).

As proved in the Lemma IX5.22 in Jacod and Shiryaev(2003), we only need prove (3.12) for $K(x, u) = g_a(x)g(x)R(u)$, where $R(u)$ is a continuous function on $\mathscr{K}$, $g \in \mathbb{C}_2^b(R)$.

As described in the Lemma IX 5.22 of Jacod and Shiryaev(2003),

$$\begin{aligned}
& Var[K * \lambda_n - (K * \lambda) \circ H_n]_t \\
\leq \ & |R(f(\sum_{i=1}^{[nt]-1} X_{n,i}))|Var[gg_a * \nu_n - gg_a * \nu]_t + |R(f(S_n(t-))) - R(f(\sum_{i=1}^{[nt]-1} X_{n,i}))| \cdot (gg_a * \nu)_t \\
\leq \ & ||R||Var[gg_a * \nu_n - gg_a * \nu]_t + ||g|||R(f(S_n(t-))) - R(f(\sum_{i=1}^{[nt]-1} X_{n,i}))| \cdot (g_a * \nu)_t
\end{aligned}$$

We can get

$$||R||Var[gg_a * \nu_n - gg_a * \nu]_t \to 0$$

by (3.13). Since $R(u)$ is a continuous function on $\mathscr{K}$, $R(u)$ is uniformly continuous on $\mathscr{K}$. For any $\varepsilon > 0$, there exists $\delta_1 > 0$, such that $|y - y'| < \delta_1 \Rightarrow |R(y) - R(y')| < \varepsilon$. Then we have

$$\begin{aligned}
& P(||g|||R(f(S_n(t-))) - R(f(\sum_{i=1}^{[nt]-1} X_{n,i}))| > \varepsilon) \\
\leq \; & P(|f(S_n(t-)) - f(\sum_{i=1}^{[nt]-1} X_{n,i}))| > \frac{\delta_1}{||g||}) \\
\leq \; & P(|X_{n,t}| > \frac{\delta_1}{||g||}) \\
\leq \; & 2\frac{\rho(\frac{\delta_1}{||g||}, \infty]}{n} \to 0
\end{aligned}$$

by the Lipschitz condition of f and (3.5).

Then

$$(||g||(R(f(S_n(t-))) - R(f(\sum_{i=1}^{[nt]-1} X_{n,i}))) \xrightarrow{P} 0. \tag{3.14}$$

Since $g_a * \nu$ is a increase deterministic measure, and (3.14) satisfies the assumption (ii) of the Lemma IX 5.22 in Jacod and Shiryaev(2003),

$$||g|||R(f(S_n(t-))) - R(f(\sum_{i=1}^{[nt]-1} X_{n,i}))| \cdot (g_a * \nu)_t \xrightarrow{P} 0.$$

We complete the proof. ■

Lemma 4 Under (3.5), we have

$$Var[B_n^1 - B^1 \circ S_n]_t \xrightarrow{P} 0 \quad \text{for } t > 0. \tag{3.15}$$

Proof. Let

$$K(x, u) = h(ux) - uh(x).$$

We obtain the lemma by Lemma 3. ■

Lemma 5 Under (3.5), we have

$$Var[C_n^{ij} - C^{ij} \circ S_n]_t \xrightarrow{P} 0 \quad \text{for } t > 0, \tag{3.16}$$

where $i, j = 1, 2$.

Proof. We only prove the case of $i = 1, j = 1$. The other cases are similar.

Although this lemma is different from Lemma 3, the method of proof is same as that of Lemma 3 through

$$\begin{aligned}
& Var[h^2(f((\sum_{i=1}^{[nt]-1} X_{n,i})x) * \nu_n(ds, dx) - h^2(f(Z_\alpha(s-)x) * \nu(ds, dx) \circ S_n]_t \\
\leq \; & |f^2(\sum_{i=1}^{[nt]-1} X_{n,i})|Var[x^2 * \nu_n - x^2 * \nu]_t + |f^2(S_n(t-)) - f^2(\sum_{i=1}^{[nt]-1} X_{n,i})| \cdot (x^2 * \nu)_t
\end{aligned}$$

$$\leq \quad CVar[x^2 * \nu_n - x^2 * \nu]_t + 2C|f(S_n(t-)) - f(\sum_{i=1}^{[nt]-1} X_{n,i})| \cdot (x^2 * \nu)_t$$

by $|h(x)| \leq |x|1_{|x|\leq 1}$.

From (3.13) and (3.14),

$$Var[h^2(f((\sum_{i=1}^{[nt]-1} X_{n,i})x) * \nu_n(ds, dx) - h^2(f(Z_\alpha(s-)x) * \nu(ds, dx) \circ S_n]_t \xrightarrow{P} 0.$$

Hence in order to prove (3.16), It suffices to show

$$Var[\sum_{s\leq t}(\int h(f(\sum_{j=1}^{[ns]-1} X_{n,j})x)\nu_n(\{s\}, dx))^2] \xrightarrow{P} 0 \tag{3.17}$$

which is equivalent to

$$Var[\sum_{s\leq t}(\int h(f(\sum_{j=1}^{[ns]-1} X_{n,j})x)\nu_n(\{s\}, dx))^2 - \sum_{s\leq t}(\int h(f(Z_\alpha(s-))x)\nu(\{s\}, dx))^2 \circ S_n]_t \xrightarrow{P} 0, \tag{3.18}$$

since $\nu(\{s\}, dx)) = 0$.

However,

$$Var[h(f((\sum_{i=1}^{[ns]-1} X_{n,i})x) * \nu_n(ds, dx) - h(f(Z_\alpha(s-)x) * \nu(ds, dx) \circ S_n]_t \xrightarrow{P} 0 \tag{3.19}$$

can implies (3.18), and the proof of (3.19) is similar to the above argument. We complete the proof. ■

Set

$$1_G * \omega_n(ds, dx) = 1_G(x, f(\sum_{i=1}^{[ns]-1} X_{n,i})x)\varphi_n(ds, dx) \text{ for } G \in \mathbb{B}^2.$$

Lemma 6 Under (3.5), we have that for $t > 0$,

$$Var[K * \omega_n - (K * \lambda) \circ S_n]_t \xrightarrow{P} 0 \tag{3.20}$$

for every continuous $K(x, u) \in \mathbb{C}_2^b(R \times \mathscr{K})$ satisfying $K(x, u) = 0$ for all $|x| \leq \delta$, $u \in \mathscr{K}$ for some $\delta > 0$.

Proof. Note that

$$|c_n| \leq E|\frac{X_1}{b_n}|1_{|X_1|\leq b_n} = \int_0^1 (P(|\frac{X_1}{b_n}| > y) - P(|\frac{X_1}{b_n}| > 1))dy \to 0.$$

For $a \neq 0$,

$$n(P(\frac{X_i}{b_n} - c_n < a) - P(\frac{X_i}{b_n} < a)) \leq nP(a - |c_n| \leq \frac{X_i}{b_n} \leq a + |c_n|) \to 0,$$

which implies

$$nP[\frac{X_1}{b_n} - c_n \in \cdot] \xrightarrow{v} \rho(\cdot) \tag{3.21}$$

by (3.5). From (3.21) and Lemma 2, we can get (3.20). ■

Remark 3 Based on the proof of Lemma 1-6, ν_n in Lemma 4 and 5 can be replaced by φ_n.

Proof of Theorem 1 Assume (3.4) with $f(x) = x$ holds. From Corollary 7.1 in Resnick (2007), we can get (3.5).

Assume that (3.5) holds. we prove (3.4). The proof will be presented in two steps.

(a) We prove the tightness of $H_n(t)$ by using Theorem VI4.18 in Jacod and Shiryaev (2003).

The functions $\alpha \rightsquigarrow B_t(\alpha), C_t(\alpha), g * \lambda_t(\alpha)$ are Skorokhod-continuous on $\mathbb{D}(R)$ since the truncation function is continuous. Then $B_n(t), C_n(t), g * \omega_n(t)$ are C-tight by Lemmas 4-6.

From (3.5),

$$\mathscr{L}(S_n(t)) \Rightarrow \mathscr{L}(Z_\alpha(t)).$$

It is means that $S_n(t)$ is tight. Note that

$$\sum_{i=1}^{[nt]} \frac{X_i}{b_n} = S_n(t) + [nt]c_n,$$

and $[nt]c_n \to \int_0^t \int h(x)\nu(ds, dx)$. Hence $\sum_{i=1}^{[nt]} \frac{X_i}{b_n}$ is tight by Proposition A.

$$\lim_{b\uparrow\infty} \limsup_n P(|x^2|1_{\{|x|>b\}} * \varphi_n(t \wedge S_n^a) > \varepsilon) = 0 \tag{3.22}$$

for all $t > 0$, $a > 0$, $\varepsilon > 0$ by the necessary part of Theorem VI4.18 in Jacod and Shiryaev (2003).

We have

$$\lim_{b\uparrow\infty} \limsup_n P(|x^2|1_{\{|x|>b\}} * \omega_n(t \wedge S_n^a) > \varepsilon) = 0$$

by (3.11), and hence $H_n(t)$ is tight.

(b) Identify the limiting process. We need to prove that if a subsequence, still denoted by $\widetilde{P}^n = \mathscr{L}(H_n)$, weakly converges to a limit $\widetilde{P}$ and the semimartingale $H(t)$ has predicable characteristics (B, C, λ) under $\widetilde{P}$, we can identify the limiting process. Since (3.1), the martingale problem $\varsigma(\sigma(X_0), X|\mathcal{L}_0, B, C, \lambda)$ has unique solution by Theorem 6.13 in Applebaum (2009).

So our work is to prove the semimartingale H has predicable characteristics (B, C, λ) under $\widetilde{P}$, in the other words, to prove

$$h(f(Z_\alpha(s-))x) * (\mu(ds, dx) - \nu(ds, dx)) \circ S_n(t),$$

$$(h(f(Z_\alpha(s-))x) * (\mu(ds, dx) - \nu(ds, dx)) \circ S_n(t))^2 - C^{11} \circ S_n(t),$$

$$g * \eta \circ S_n(t) - g * \lambda \circ S_n(t) \text{ for } g \in C^1(R)$$

are local martingales under $\widetilde{P}$.

Since

$$\begin{aligned} &\int_0^t \int h(f(\sum_{j=1}^{[ns]-1} X_{n,j})x)(\zeta_n(ds, dx) - \varphi_n(ds, dx)) - h(f(Z_\alpha(s-))x) * (\mu(ds, dx) - \nu(ds, dx)) \circ S_n(t) \\ = \;& h(f((\sum_{i=1}^{[nt]-1} X_{n,i})x) * \varphi_n(ds, dx) - h(f(Z_\alpha(s-)x) * \nu(ds, dx) \circ S_n(t), \end{aligned}$$

(3.19), Lemma 6 and Remark 3 implies

$$\int_0^t \int h(f(\sum_{j=1}^{[ns]-1} X_{n,j})x)(\zeta_n(ds,dx)-\varphi_n(ds,dx))-h(f(Z_\alpha(s-))x)*(\mu(ds,dx)-\nu(ds,dx))\circ S_n(t) \xrightarrow{P} 0. \tag{3.23}$$

Set

$$\widetilde{C}_n^{11}(t)=\int_0^t \int h^2(f(\sum_{j=1}^{[ns]-1} X_{n,j})x)\varphi_n(ds,dx)-\sum_{s\le t}(\int h(f(\sum_{j=1}^{[ns]-1} X_{n,j})x)\varphi_n(\{s\},dx))^2,$$

since

$$\mathscr{L}(S_n(t))\Rightarrow \widetilde{P},$$

(3.11) implies that $C^{11}\circ S_n(t)\le C$, and Lemma 5, 6 implies that $P(\widetilde{C}_n^{11}(1)\ge C+1)\to 0$ as $n\to\infty$. Set $T_n=\inf\{t:\widetilde{C}_n^{11}(t)>C+1\}$, we have

$$\lim_{n\to\infty} P(T_n<1)=0.$$

so

$$E(\sup_{0\le t\le 1}|\int_0^{t\wedge T_n}\int h(f(\sum_{j=1}^{[ns]-1} X_{n,j})x)(\zeta_n(ds,dx)-\varphi_n(ds,dx))|^2)\le 4E(\widetilde{C}_n^{11}(T_n)) \tag{3.24}$$

by Doob's inequality.

Since

$$\int_0^t \int h(f(\sum_{j=1}^{[ns]-1} X_{n,j})x)(\zeta_n(ds,dx)-\varphi_n(ds,dx))$$

is a local martingale, (3.23) and (3.24) imply that

$$h(f(Z_\alpha(s-))x)*(\mu(ds,dx)-\nu(ds,dx))\circ S_n(t)$$

is a local martingale under $\widetilde{P}$ by Proposition B.

A simple computation obtain that

$$\begin{aligned}
&(\int_0^t \int h(f(\sum_{j=1}^{[ns]-1} X_{n,j})x)(\zeta_n(ds,dx)-\varphi_n(ds,dx)))^2\\
&-(h(f(Z_\alpha(s-))x)*(\mu(ds,dx)-\nu(ds,dx))\circ S_n(t))^2+C^{11}\circ S_n(t)-\widetilde{C}_n^{11}(t)\\
=\;&(\int_0^t \int h(f(\sum_{j=1}^{[ns]-1} X_{n,j})x)(\zeta_n(ds,dx)-\varphi_n(ds,dx))\\
&\cdot(\int_0^t \int h(f(\sum_{j=1}^{[ns]-1} X_{n,j})x)(\zeta_n(ds,dx)-\varphi_n(ds,dx))-h(f(Z_\alpha(s-))x)*(\mu(ds,dx)-\nu(ds,dx))\circ S_n(t)))\\
&+(h(f(Z_\alpha(s-))x)*(\mu(ds,dx)-\nu(ds,dx))\circ S_n(t)))\\
&\cdot(\int_0^t \int h(f(\sum_{j=1}^{[ns]-1} X_{n,j})x)(\zeta_n(ds,dx)-\varphi_n(ds,dx))-h(f(Z_\alpha(s-))x)*(\mu(ds,dx)-\nu(ds,dx))\circ S_n(t))\\
&+C^{11}\circ S_n(t)-\widetilde{C}_n^{11}(t).
\end{aligned}$$

We have that

$$\int_0^{t\wedge T_n}\int h(f(\sum_{j=1}^{[ns]-1}X_{n,j})x)(\zeta_n(ds,dx)-\varphi_n(ds,dx))$$

is uniformly integrable by (3.24), thus

$$\begin{aligned}&(\int_0^t\int h(f(\sum_{j=1}^{[ns]-1}X_{n,j})x)(\zeta_n(ds,dx)-\varphi_n(ds,dx)))^2\\&-(h(f(Z_\alpha(s-))x)*(\mu(ds,dx)-\nu(ds,dx))\circ S_n(t))^2+C^{11}\circ S_n(t)-\widetilde{C}_n^{11}(t)\xrightarrow{P}0\end{aligned} \tag{3.25}$$

by (3.23), Lemma 5 and Remark 3.

By Lemma VII 3.34 of Jacod and Shiryaev (2003),

$$E(\sup_{0\le t\le 1}|\int_0^{t\wedge T_n}\int h(f(\sum_{j=1}^{[ns]-1}X_{n,j})x)(\zeta_n(ds,dx)-\varphi_n(ds,dx))|^4)\le K_1[E(\widetilde{C}_n^{11}(T_n))^2]^{\frac{1}{2}}+K_2E(\widetilde{C}_n^{11}(T_n))^2 \tag{3.26}$$

where K_1 and K_2 are constants.

Since

$$(\int_0^t\int h(f(\sum_{j=1}^{[ns]-1}X_{n,j})x)(\zeta_n(ds,dx)-\varphi_n(ds,dx)))^2-\widetilde{C}_n^{11}(t)$$

is a local martingale, (3.25) and (3.26) implies

$$(h(f(Z_\alpha(s-))x)*(\mu(ds,dx)-\nu(ds,dx))\circ S_n(t))^2-C^{11}\circ S_n(t)$$

is local martingale under $\widetilde{P}$ by Proposition B.

For

$$g*\eta\circ S_n(t)-g*\lambda\circ S_n(t)\text{ for }g\in C^1(R),$$

we can get the similar conclusion by Lemma 6. We complete the proof. ■

The proof of Theorem 2 is similar except minor changes, we omit it here.

4 Convergence to Stochastic Integral Driven by Lévy $\alpha-$Stable Process: The Multivariate Case.

In this section, we use the similar method to obtain the weak convergence for various general functionals of partial sums of i.i.d. heavy-tailed random vectors. Since the idea and method is similar, the proof are not given.

Theorem 3 Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous differentiable function such that

$$|f(x)-f(y)|\le K|x-y|^a$$

for some constants $K>0$, $a>0$ and all $x,y\in\mathbb{R}$. Suppose that $\{\xi_n\}_{n\ge1}:=\{(\xi_n^1,\xi_n^2)\}_{n\ge1}$ are i.i.d. random vectors. Set

$$\xi_{n,j}=\frac{\xi_j}{b_n}-E(h(\frac{\xi_j}{b_n}))$$

for some $b_n \to \infty$. Then

$$(\sum_{i=1}^{[nt]} \xi_{n,i}, \sum_{i=2}^{[nt]} f(\sum_{j=1}^{i-1} \xi_{n,j}^1)\xi_{n,i}^2) \Rightarrow (Z_\alpha(t), \int_0^t f(Z_\alpha^1(s-))dZ_\alpha^2(s))$$

in $\mathbb{D}[0,1]$, where $Z_\alpha(s)$ is a 2-dimesional $\alpha-$stable Lévy Process with Lévy measure ν iff ξ_1 is a random vector satisfying the usual multivariate regular variation condition with exponent α and

$$nP[\frac{\xi_1}{b_n} \in \cdot] \xrightarrow{v} \nu(\cdot) \tag{4.1}$$

in $M_p(E_2)$, where $E_2 = [-\infty, \infty] \setminus \{0\} \otimes [-\infty, \infty] \setminus \{0\}$.

5 Discussion.

In this paper, we use a continuous function $h(x)$ for technical convenience. In fact, we can take $h(x) = x1_{|x|\leq 1}$ to replace the continuous function through small change.

We only discuss independent random variables. It will be more complex for dependence case. Recently, a lot of authors discussed the functional limit theorems for

$$X_n(t) = \frac{1}{b_n} \sum_{i=1}^{[nt]-1} X_i - tc_n$$

under dependence assumption (see Balan and Louhichi (2009), Tyran-Kkamińska(2010 a,b)). They employed the point process method to deal with dependence. We hope that the method used in this paper will be useful for study of the dependence heavy-tail random variables.

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